Some Application Notes on Reduction Operators

T. Tuma
University of Ljubljana,
Faculty of Electrical Engineering,
1000 Ljubljana, Tržaška 25, Slovenia.
Phone: 386-61-1768-329, Fax: 386-61-1264-630,
Email: tadej.tuma@fe.uni-lj.si

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1 Introduction

In earlier publications [3, 4] the theoretical background of reduction operators has been thoroughly discussed. This letter presents an interesting case study of the reduction operator approach to circuit equations. Based on an extremely simple example circuit, the relation between the nodal equations and the state variable equations is examined.

2 A review of the reduction operator

Let R_n be the reduction operator of rank n. Its formal definitions including all properties and proofs can be found in [3, 4]. For convenience though, we will repeat its most essential definition

$$R_n(A) \triangleq \begin{bmatrix} I_{n\times n} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}_{r\times r} \begin{bmatrix} A_{11_{n\times n}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{r\times c} = \begin{bmatrix} A_{11_{n\times n}} & A_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}_{r\times c} \triangleq \tilde{A}_U, \tag{1}$$

where

$$\tilde{A}_{22} \triangleq A_{22} - A_{21}A_{11}^{-1}A_{12}. \tag{2}$$

As can be seen from (1), the reduction operator of rank n reduces all elements in the $(r-n)\times n$ south–western corner of A to zero, transforming it to an upper block triangular form $\tilde{A}_U$ [1]. A may be any matrix with more than n rows and columns, provided that there is a non-sigular square submatrix A_{11} of order n in its north–western corner. The application of the reduction operator R_n actually implies the inversion of the north–western portion in A. Complementary to the reduction operator, we further define the *transposed* reduction operator

$$\begin{aligned} R_n^T(A) &\triangleq \left(R_n(A^T)\right)^T = \left(\begin{bmatrix} I & 0 \\ -A_{12}^T A_{11}^{T\,-1} & I \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\right)^T = \\ &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & \tilde{A}_{22} \end{bmatrix} \triangleq \tilde{A}_L, \end{aligned} \tag{3}$$

where $\tilde{A}_{22}$ is identically equal to (2). As expected, the transposed reduction operator transforms matrix A into a lower block triangular form $\tilde{A}_L$ [1].

R_m and its close relative R_n^T are then used together with three general permutation matrices P, Q and R to constitute a general operator equation, mapping any matrix equation

$$AB = 0 \tag{4}$$

into

$$R_m(PAQ^T)\, R_n^T(QBR^T) \triangleq R_m(\hat{A})\, R_n^T(\hat{B}) \triangleq \tilde{\hat{A}}\tilde{\hat{B}} = 0. \tag{5}$$

It has been established [3, 4], that the repeating application of this operator equation suffices to describe entire circuit analysis approaches. For instance, a series of reduction operators can be used to execute a Gaussian forward elimination, a back substitution, a triangular decomposition or a Gauss–Jordan elimination. Also, a circuit's nodal, cutset or loop equation set may be formulated solely by using reduction operators and permutation matrices.

3 The application example

Consider the following extremely simple two–node ($n = 2$) circuit in figure 1. It is composed of four branches ($b = 4$), namely one capacitor branch ($c = 1$), three resistive branches ($r = 3$) and no inductive branch ($l = 0$).

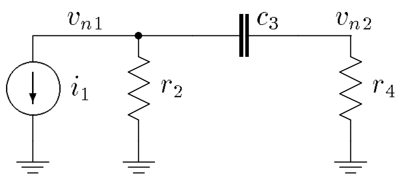

Figure 1: The sample circuit

This circuit requires a basic tableau (6) of 10th order [2]. Since we intend to apply reduction operators to the tableau, we use a rather unusual notation (6). The equation right hand side has been appended to the coefficient matrix on the left hand side, leaving a zero matrix behind. This notation also necessitates an additional -1 constant element in the variable vector. The blank areas in (6) correspond to zero matrices and should be considered full of zero coefficients.

The tableau in (6) is now suitable for reduction operators, since it has the required form (4). The matrix in (6), containing derivative operators, actually makes (6) a linear differential equation set rather than a linear one. In order to produce the correct nodal equations, we must replace all reactances in the circuit by adequate linear time invariant models. In our case, the capacitor branch equation $\frac{d}{dt}v_{b3} - i_{b3} = 0$ becomes $g_3 v_{b3} - i_{b3} = i_3$. The current excitation $i_1(t)$ should be also considered time independent i_1.

$$
\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & & & & & \\
1 & 0 & 0 & -1 & 0 & 0 & & & & & \\
1 & -1 & 0 & 0 & -1 & 0 & & & & & \\
0 & 1 & 0 & 0 & 0 & -1 & & & & & \\
 & & & & & & 1 & 1 & 1 & 0 & \\
 & & & & & & 0 & 0 & -1 & 1 & \\
 & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i_1(t) \\
 & & 0 & -1 & 0 & 0 & 0 & r_2 & 0 & 0 & 0 \\
 & & 0 & 0 & \frac{d}{dt} & 0 & 0 & 0 & -\frac{1}{c_3} & 0 & 0 \\
 & & 0 & 0 & 0 & -1 & 0 & 0 & 0 & r_4 & 0
\end{bmatrix}
\begin{bmatrix} v_{n1} \\ v_{n2} \\ v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ -1 \end{bmatrix} = 0. \tag{6}
$$

At this stage the entire nodal analysis approach including Gaussian elimination and extended back substitution can be performed by a simple series of reduction operators in combination with some permutation matrices

$$
\underbrace{R_2(R_1(P_2\, \underbrace{R_9(\underbrace{R_8(R_4(P_1 \underbrace{\begin{bmatrix} A^T & -I & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & Y & Z & e \end{bmatrix}}_{\text{Tableau (6)}} Q_1^T)))}_{\text{Nodal equations (9)}}\, Q_2^T))}_{\text{Gaussian elimination (10)}}}_{\text{Extended back substitution}}(Q_2 Q_1 \begin{bmatrix} v_n \\ v_b \\ i_b \\ -1 \end{bmatrix}) = 0. \tag{7}
$$

Let us discuss (7) gradually. The first step consists of transforming the basic circuit tableau to a nodal equation set. This is done in by the permutation matrices

$$
P_1 = \begin{bmatrix} I_{4\times 4} & 0 & 0 \\ 0 & 0 & I_{4\times 4} \\ 0 & I_{2\times 2} & 0 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0 & I_{4\times 4} & 0 & 0 \\ 0 & 0 & I_{4\times 4} & 0 \\ I_{2\times 2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

and the reduction operators R_4, R_8. The result can be seen in

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 & & & & & 1 & 0 & \\
0 & -1 & 0 & 0 & & & & & 1 & 0 & \\
0 & 0 & -1 & 0 & & & & & 1 & -1 & \\
0 & 0 & 0 & -1 & & & & & 0 & 1 & \\
 & & & & 1 & 0 & 0 & 0 & 0 & 0 & i_1 \\
 & & & & 0 & r_2 & 0 & 0 & -1 & 0 & 0 \\
 & & & & 0 & 0 & -1 & 0 & g_3 & -g_3 & i_3 \\
 & & & & 0 & 0 & 0 & r_4 & 0 & -1 & 0 \\
 & & & & & & & & \frac{1}{r_2} + g_3 & -g_3 & i_3 - i_1 \\
 & & & & & & & & -g_3 & \frac{1}{r_4} + g_3 & -i_3
\end{bmatrix}
\begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ v_{n1} \\ v_{n2} \\ -1 \end{bmatrix} = 0. \tag{9}
$$

The second order nodal equation set is only a small part of (9), namely the last two rows. The triangular equation set above it is relating the independent nodal voltages v_{n1}, v_{n2} to all the other circuit variables.

In our second order equation set the Gaussian elimination's forward course consists of a single reduction operator i.e. R_9 in (7). In this way only the last row of (9) is affected

$$\begin{bmatrix} & \vdots & \vdots & \vdots \\ \cdots & \frac{1}{r_2}+g_3 & -g_3 & i_3-i_1 \\ \cdots & 0 & \frac{1+r_4 g_3}{r_4}-\frac{r_2 g_3^2}{1+r_2 g_3} & \frac{r_2 g_3 (i_3-i_1)}{1+r_2 g_3}-i_3 \end{bmatrix} \begin{bmatrix} \vdots \\ v_{n1} \\ v_{n2} \\ -1 \end{bmatrix} = 0. \tag{10}$$

A second order back substitution should produce the circuit nodal voltages from the last two rows in (10). Considering the upper triangular matrix portion above the nodal equations however, we can run an extended 10th order back substitution, assembling the values of all circuit variables.

Using reduction operators [4], any back substitution is formally executed as a forward substitution of the reversed matrix order. The reversing matrices in (7) are antidiagonal unit matrices P_2 and Q_2. The substitution itself needs only the two reduction operators R_1 and R_2, in order to arrive at the final diagonal form, from which the solution vector is obtained by division.

Let us now go back to the circuit tableau (6) and use the state variable approach. This time the derivation operators are kept in the equation set rather than being replaced with time invariant models. The reduction operator expression for state variable analysis yields a remarkable resemblance between the nodal equations (7) and the state equations

$$\underbrace{R_4(R_1(P_2\, \underbrace{R_9(R_6(R_3(P_1 \underbrace{\begin{bmatrix} A^T & -I & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & Y(t) & Z(t) & e(t) \end{bmatrix}}_{\text{Tableau (6)}} Q_1^T)))\, Q_2^T}_{\text{State equations (13)}}))}_{\text{Output equations}}(Q_2 Q_1 \begin{bmatrix} v_n \\ v_b \\ i_b \\ -1 \end{bmatrix}) = 0. \tag{11}$$

In order to clarify (11) we will explicitly discuss its three stages. First, a permutation is needed to determine the circuits order of complexity and to select the state variables. This is done by heuristic methods and leads to the permutation matrices

$$P_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}. \tag{12}$$

Secondly, three reduction operators R_3, R_6 and R_9 are applied in (11). The permuted equation set is transformed to an upper block triangular form with the state equations in the last block

$$\begin{bmatrix} -1 & 0 & 0 & & & & 1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & & & 1 & 0 & 0 & & 0 \\ 0 & 0 & -1 & & & & 0 & 1 & 0 & & 0 \\ & & & 1 & 0 & 0 & 0 & 0 & 0 & & i_1(t) \\ & & & 0 & r_2 & 0 & -1 & 0 & 0 & & 0 \\ & & & 0 & 0 & r_4 & 0 & -1 & 0 & & 0 \\ & & & & & & \frac{1}{r_2} & 0 & 1 & 0 & -i_1(t) \\ & & & & & & 0 & \frac{1}{r_4} & -1 & 0 & 0 \\ & & & & & & 1 & -1 & 0 & -1 & 0 \\ & & & & & & & & & \frac{d}{dt} - \Psi & \psi(t) \end{bmatrix} \begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b4} \\ i_{b1} \\ i_{b2} \\ i_{b4} \\ v_{n1} \\ v_{n2} \\ i_{b3} \\ v_{b3} \\ -1 \end{bmatrix} = 0, \qquad (13)$$

where

$$[\, -\Psi \quad \psi(t) \,] = -[\, 0 \quad 0 \quad -\tfrac{1}{c_3} \,] \begin{bmatrix} \frac{1}{r_2} & 0 & 1 \\ 0 & \frac{1}{r_4} & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -i_1(t) \\ 0 & 0 \\ -1 & 0 \end{bmatrix} = [\, \tfrac{1}{c_3(r_2+r_4)} \quad \tfrac{-i_1(t) r_2}{c_3(r_2+r_4)} \,]. \qquad (14)$$

As expected, we find in this simple case only one state equation, involving the branch voltage across our only capacitor v_{b3}. The modified nodal equations of the circuit's resistive portion can be easily identified directly above the state equation. In order to compute the state equation coefficients Ψ and ψ the inversion of this nodal equation set is compulsory (14). The final first order differential equation in the last row of (13) hence reads

$$\frac{d}{dt} v_{b3} = \frac{-1}{c_3(r_2 + r_4)}\, v_{b3} + \frac{-i_1(t) r_2}{c_3(r_2 + r_4)}. \qquad (15)$$

It isn't until now, that the circuits time dependence is resolved by an integration algorithm. Once the circuit transient response of the state variables is determined, the output equations above the state equation in (13) are employed to get the values of all the other circuit variables. This is done by a simple block–wise back substitution, similar to the one in (7). Again an antidiagonal permutation P_2, Q_2 is used to reverse the equation order. Then the last two reduction operators R_1, R_4 from (11) take effect and obtain the values of the output variables.

At this point, perhaps some application notes are necessary in order to explain the inclusion of time derivative operators (6) into a computer data structure. Today most programming tools are equipped with complex arithmetics, which is indispensable for frequency domain analysis anyway. By using complex arithmetics the time derivative operators in (6) can be accommodated as unit imaginary parts of coefficients. All subsequent operations are simply performed in a complex domain.

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